Math 600 Day 14: Homotopy Invariance of de Rham Cohomology

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Differential forms on manifolds. Let M^n be a smooth manifold. A **differential p-form** ω on M is a choice of a p-form $\omega(x) \epsilon \Lambda^p T_x M$ for each $x \epsilon M$.

If (f, U) is a coordinate system on an open subset of M, then there is a unique differential p-form ω_U on U such that $f^*(\omega(f(u))) = \omega_U(u)$ for each point $u \epsilon U$.

If the differential p-forms ω_U are differentiable for a family of coordinate systems which cover M, then the differential p-form ω on M is said to be differentiable (or smooth) , typically of class C^∞.

This definition does not depend on the choice of coordinate systems covering M.

Exterior derivatives.

Given a smooth differential p-form ω on the smooth manifold M^n, there is a unique smooth differential $(p+1)$-form $d\omega$ on M such that for every coordinate system (f, U) we have

$$f^*(d\omega) = d(f^*\omega).$$

Let $\Omega_p(M)$ denote the vector space of smooth p-forms on the smooth k-manifold M, and

$$\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus ... \oplus \Omega^k(M)$$

the **differential graded algebra** of smooth forms on M.

Let M^m be a smooth m-manifold, with or without boundary. Let $\Omega^k(M)$ denote the vector space of smooth k-forms on M, for $0 \leq k \leq m$. These vector spaces are connected by exterior differentiation:

$$\Omega^0(M) - d \rightarrow \Omega^1(M) - d \rightarrow ... - d \rightarrow \Omega^m(M).$$

Since $d^2 = 0$, the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$ is a subspace of the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

The corresponding quotient space,

$$H^k_{deR}(M) = \frac{ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{im(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

is the k^{th} **de Rham cohomology group** of M (actually a real vector space). Thus $H^k_{deR}(M)$ measures the extent to which closed k-forms on M can fail to be exact.

The above repeats for a smooth manifold M what we have already said for an open subset U of Euclidean space.

Putting the vector spaces $\Omega^k(M)$ together into one package, we get

$$\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus ... \oplus \Omega^k(M),$$

the differential graded algebra of smooth forms on M. The multiplication comes from the exterior product

$$\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M).$$

Note that

1. $d\phi_1 = 0$ and $d\phi_2 = 0$ implies $d(\phi_1 \wedge \phi_2) = 0$.
   That is, *closed* $\wedge$ *closed* $=$ *closed*.
2. $\phi_1 = d\mu_1$ and $d\phi_2 = 0$ implies $\phi_1 \wedge \phi_2 = d(\mu_1 \wedge \phi_2)$.
   That is, *exact* $\wedge$ *closed* $=$ *exact* .

Hence the exterior product at the level of differential forms induces a cup product at the level of de Rham cohomology:

$$\cup : H^p_{deR}(M) \times H^q_{deR}(M) \to H^{p+q}_{deR}(M)$$

This cup product makes

$$H^*_{deR}(M) = H^0_{deR}(M) \oplus H^1_{deR}(M) \oplus \ldots \oplus H^m_{deR}(M)$$

into a graded algebra, called the **de Rham cohomology algebra of M**.

Induced mappings.

A smooth map $f : M \longrightarrow N$ between smooth manifolds induces maps in the other direction, $f^* : \Omega^k(N) \rightarrow \Omega^k(M)$ between smooth k-forms. These induced maps commute with exterior products,

$$f^*(\phi_1 \wedge \phi_2) = f^*(\phi_1) \wedge f^*(\phi_2),$$

and hence assemble to a homomorphism of graded algebras

$$f* : \Omega^*(N) \rightarrow \Omega^*(M).$$

If ϕ is a closed k-form on N, then $f^*(\phi)$ is a closed k-form on M because $d(f^*\phi) = f^*(d\phi) = f^*(0) = 0$.

If ϕ is an exact k-form on N, say $\phi = d\mu$, then $f^*(\phi)$ is an exact k-form on M because $d(f^*\mu) = f^*(d\mu) = f^*(\phi)$.

Hence we get an induced map $f^* : H^k_{deR}(N) \to H^k_{deR}(M)$, defined by $f^*([\phi]) = [f^*\phi]$, where $[\phi]$ represents the cohomology class of the closed k-form ϕ. These induced linear maps assemble to an algebra homomorphism

$$f* : H^*_{deR}(N) \to H^*_{deR}(M)$$

between the graded de Rham cohomology algebras.